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THE INTERACTION BETWEEN SHOCK AND RAREFACTION WAVES IN THE PROBLEM OF ANGULAR PISTON*

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Motion of a dihedral piston one of whose faces moves into while the other pulls out from the initially quiescent perfect gas is considered. The problem is investigated analytically in the linearized self-similar formulation, and numerically in totally nonlinearized formulation. Analytic form of the basic shock wave is obtained on the assumption that the velocity of the moving-in face is small in comparison with the speed of sound. The method of finite differences is applied to different velocities of piston faces. The pattern of arising flows is investigated.

The potential flows generated by an angular piston with flat faces pulling out from a perfect gas were considered in /1 – 7/ (the piston moving into an isothermal gas was considered in /1/). When the velocity of the piston faces moving out exceeds some critical value the gas becomes detached from the piston, and there is flow into vacuum. An exact solution defining a particular case of such flow was considered in /2/. The pulling out of an angular piston was considered in /3,4/ for faces velocities below the critical. In /5,6/ flows generated by the pulling out of a trihedral angle was investigated. In the perturbed region at the corner apex the flow is of the form of potential triple wave. Approximate solution defining the move into and pulling out motions of a dihedral piston with apex angle close to π were obtained in /7 – 9/ in linear formulation of the problem. The two-dimensional disintegration of discontinuity of such form was also considered.

Unlike in the majority of cited publications, a nonisentropic flow whose potentiallity is not assumed is considered here. When the velocity of the pulling out face exceeds some critical value, a vacuum zone appears near it. It is not a priori clear whether in the moving-in piston face neighborhood the plane gas-vacuum interface becomes curved, or whether tongues of gas can penetrate there the vacuum region.

Let a perfect polytropic gas (the speed of sound $c_0 = (\gamma P^\circ/\rho^\circ)^{1/2}$) is at rest inside a dihedral angle $x_1 = 0, x_2 = 0$. At $t = 0$ the dihedral piston begins to move in a plane-parallel uniform motion at constant speed of the apex $U_0 = (a, b)$, $a < -1/\varkappa$, $\varkappa = (\gamma - 1)/2$, where $b > 0$ is a small quantity. The generated plane flow depends on a single dimensionless parameter b/c° (when $a < -1/\varkappa$ the piston face $x_1 = at$ is separated from the gas by the vacuum zone and does not affect the flow). We assume the flow to be self-similar, and all quantities depend on $\xi_1 = x_1/t$, $\xi_2 = x_2/t$. In what follows we assume $c^\circ = 1$.

In Fig.1 the flow is represented in the ξ_1, ξ_2 plane. Region *1* bounded by the straight lines $\xi_1 = 1$, $\xi_2 = D = (\gamma + 1)\, b/4 + [1 + ((\gamma + 1)\, b/4)^2]^{1/2}$, is quiescent. In region *3* the motion is uniform, the velocity components are $u_1 = 0$, $u_2 = b$, and the speed of sound $c = c_2 > 1$. In region *2* defined by the inequalities $-1/\varkappa < \xi_1 < 1$, $\xi_2 > f(\xi_1)$ ($\xi_2 = f(\xi_1)$ is the shock wave) we have a Riemann rarefaction wave, and

$$u_2 = 0,\ u_1 = u_0(\xi_1) = 2(\xi_1 - 1)/(\gamma + 1),\ c = c_0(\xi_1) = 1 + \varkappa u_0(\xi_1).$$

Gasdynamic quantities throughout the region of the considered here self-similar flow is defined by the system of equations

$$Lu_k + \frac{c}{\varkappa}\left(\frac{\partial c}{\partial \xi_1} - c\frac{\partial \Psi}{\partial \xi_1}\right) = 0;\quad k = 1, 2. \tag{1}$$
$$Lc = \varkappa c\left(\frac{\partial u_1}{\partial \xi_1} + \frac{\partial u_2}{\partial \xi_2}\right) = 0$$
$$L\Psi = 0;\ L = (u_1 - \xi_1)\frac{\partial}{\partial \xi_1} + (u_2 - \xi_2)\frac{\partial}{\partial \xi_2},\quad \Psi(\xi_1, \xi_2) = \frac{1}{\gamma}\ln\frac{P}{\rho^\gamma}$$

where $c(\xi_1, \xi_2)$ is the speed of sound.

Let us first construct an approximate solution of the stated problem by using linearizing system (1).

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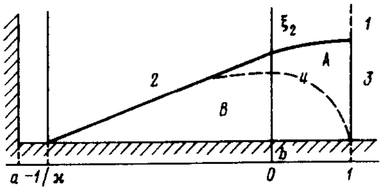

Fig.1

If the piston face velocity b is considered as the small parameter we can consider the shock wave $\xi_2 = f(\xi_1)$ as weak and disregard the entropy variation. We represent the gasdynamic quantities in region 4 in the form

$$u_1 = \frac{2}{\gamma+1}(\xi_1 - 1) + bU(\xi_1, \xi_2) = u_0(\xi_1) + bU, \qquad u_2 = bV(\xi_1, \xi_2) \tag{2}$$
$$c = \frac{\gamma-1}{\gamma+1}\xi_1 + \frac{2}{\gamma+1} + bC(\xi_1, \xi_2) = c_0(\xi_1) + bC$$

We substitute (2) into (1), carry out linearization, and obtain for U, V, C the system of three equations

$$L_0 U - c_0 \frac{\partial C}{\partial \xi_1} = (U + C)\frac{\partial U_0}{\partial \xi_1} = \frac{2}{\gamma+1}(U + C); \quad L_0 V - c_0 \frac{\partial C}{\partial \xi_2} = 0 \tag{3}$$
$$L_0 C - \varkappa c_0 \left(\frac{\partial}{\partial \xi_1} + \frac{\partial V}{\partial \xi_2}\right) = (U + C)\frac{\partial c_0}{\partial \xi_1} = \frac{\gamma-1}{\gamma+1}(U + C);$$
$$L_0 = c_0 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}$$

where u_0, c_0 are functions of ξ_1, not constants as in /8,9/, which results in a more complex structure of the system of equations, and necessitates a change in the method of the approximate solution derivation.

In region 4 system (3) is hyperbolic when $C_0 > 0$ and has three sets of characteristics along which are satisfied the following differential relations:

$$1^\circ. \quad \xi_1 = \text{const}; \quad \xi_2\, dU - c_0\, dV - \frac{\xi_2}{\varkappa}\, dC = \frac{4}{\gamma+1}(U + C)\, d\xi_2$$

$$2^\circ. \quad \xi_2 = \text{const}\, c_0^{\alpha}, \quad \alpha = \frac{\gamma+1}{\gamma-1};$$
$$c_0\, dU + \xi_2\, dV - \frac{c_0}{\varkappa}\, dC = \frac{2}{\gamma+1}(U + c)\, d\xi_2$$

$$3^\circ. \quad \xi_2 = c_0\left(\text{const}\, c_0^{\beta} + \frac{\gamma+1}{3-\gamma}\right)^{1/2}, \quad \beta = \frac{3-\gamma}{\gamma-1};$$
$$-\xi_2^2\, dU + \xi_2 c_0\, dV + \frac{\xi_2^2}{\varkappa}\, dC = \frac{2}{\gamma+1} c_0 (U + V)\, d\xi_2$$

The Hugoniot conditions on a weak shock wave $\xi_2 = f(\xi_1)$ are of the form /10/

$$((\mathbf{V}_1 - \mathbf{V}_0)\,\mathbf{N})^2 = \frac{2}{\gamma+1}\left[\frac{c_0^2}{((\mathbf{V}_0 - \mathbf{U}_f)N)} - ((\mathbf{V}_0 - \mathbf{U}_f)\,\mathbf{N})\right]((\mathbf{V}_1 - \mathbf{V}_0)\,\mathbf{N}) \tag{4}$$
$$(\mathbf{V}_1\mathbf{T}) = (\mathbf{V}_0\mathbf{T}), \quad c_1^2 = c_0^2 + \varkappa\,[((\mathbf{V}_0 - \mathbf{U}_f)\,\mathbf{N})^2 - ((\mathbf{V}_1\mathbf{U}_f)\,\mathbf{N}^2)]$$
$$V_0 = (u_0(\xi_1), 0), \quad \mathbf{V}_1 = (u_1, u_2), \quad \mathbf{T} = (1, f')/(1 + f'^2)^{1/2}$$
$$\mathbf{N} = (f', -1)/(1 + f'^2)^{1/2}, \quad \mathbf{U}_f = (\xi_1, f(\xi_1))$$

where $\mathbf{V}_0$ is the velocity vector of gas particles ahead of the shock wave, and $\mathbf{V}_1$ and C_1 are respectively, the vector of gas particle velocity vector and the speed of sound behind the shock wave.

Substituting in conditions (4) for u_1, u_2, c their approximation (2) and linearizing the resulting equations, we obtain conditions for U, V, C and an ordinary differential equation for the determination of the approximate form of shock wave $\xi_2 = f_0(\xi_1)$. Its solution yields

$$f_0(\xi_1) = c_0\left(\text{const}\, c_0^{\beta} + \frac{\gamma+1}{3-\gamma}\right)^{1/2}, \quad \beta = \frac{3-\gamma}{\gamma-1} \tag{5}$$
$$C(\xi_1, f_0(\xi_1)) = \varkappa\,(1 + f_0'^2)^{1/2} V(\xi_1, f_0(\xi_1))$$
$$U(\xi_1, f_0(\xi_1)) = -f_0'\, V(\xi_1, f_0(\xi_1))$$

The shock wave approximation $\xi_2 = f_0(\xi_1)$ is a characteristic of system (3). Substituting (5) into the differential relation on the characteristic of set 3°, we obtain

$$\frac{dV(\xi_1)}{d\xi_1} = V(\xi_1)\frac{c_0}{f_0(c_0^2 + f_0^2)}\left[\frac{2}{\gamma+1} c_0 (\varkappa(1 + f_0'^2)^{1/2} - f_0') - f_0^2 f_0'' - \frac{f_0^2 f_0' f_0''}{(1 + f_0^2)^{1/2}}\right] \tag{6}$$

The first of Eq.(5) defines the form of the curvilinear shock wave. Integrating Eq.(6) along it and substituting $V(\xi_1)$ into the second and third of Eqs.(5), we find that along the curvilinear shock wave $V(\xi_1), U(\xi_1), C(\xi_1)$.

The nonisentropic flow in region 4 is separated from that in region 3 by the weak shock wave $\xi_1 = g(\xi_2)$. Substituting Eqs.(2) into the Hugoniot conditions (4), where $U_0(0, b)$ $c_0 = c_2 = 1$ — const and linearizing, we obtain the approximate form of the shock wave $\xi_1 = g_0 = 1$ which is a characteristic of system (3). The linearized conditions (4) with the differential relation along the characteristic of set 1° yield for U, V, C along $\xi_1 = 1$ the expressions

$$V = 1, \quad C = \varkappa\,(1 + f_0'^2\,(1))^{1/2} = c_3 - \text{const}, \quad U = (c_3 - f_0'(1))(\xi_2/D)^\delta - c_3, \quad \delta = 4/\gamma + 1 \qquad (7)$$

Three shock waves converge at point $(1, D)$. This configuration of only three shock waves is incompatible with /11/, since one more strong discontinuity, usually a contact one, must pass through this point, however we shall not take it into account.

Boundary conditions for system (3) are: (5) and (6) along $\xi_2 = f_0(\xi_1) \times (V(1) = 1)$, (7) along $\xi_1 = 1$, and $V = 1$ along $\xi_2 = b$. Boundary conditions along $\xi_2 = f_0(\xi_1)$ and $\xi_1 = 1$ define the data for the Goursat problem which is solved in region A (Fig.1). Then the characteristic Cauchy problem is solved in region B bounded by the characteristic of set 3° and the piston. The boundary value problem in region 4 (Fig.1) was obtained using the method of characteristics.

The profiles of quantities U, V, C are shown in Fig.2 for the cross sections $\xi_1 = \text{const}$, with curves $0, 1, 2, 3$ corresponding to values $\xi_1 = 1;\ 0.284;\ -1.977;\ -3.747$ $(b = 0.2)$ and calculated for $\gamma = 1.4$.

One of the difficulties of obtaining by this method of an approximate solution for $-1/\varkappa < a < 0$ is related to the emergence in the system of approximate equations of a region of ellipticity near the angle apex.

Let us consider the motion of the angular piston for $-1 < \varkappa < a < 0$ and arbitrary (but not small) $b > 0$. The vacuum zone near the piston is now absent, and the flow depends on two dimensionless parameters a/c_0 and b/c_0, and the generated shock waves are of arbitrary intensity. For analyzing this problem a method of difference calculation was developed using a specially produced program. This method was also used for checking the assumption of solution self-similarity and for determining the region approximate solution applicability. The unsteady plane problem of uniform plane-parallel motion of an angular piston starting from rest was calculated without the assumption of its self-similarity (Fig.3). The corner apex velocity is $U_0 = (a, b)$, $-1/\varkappa < a < 0$. Near the piston face $x_1 = at$ appear new regions, viz. region 5 of constant flow $u_1 = 0$, $u_2 = a$, $c = c_3 = \text{const}$ and 6 in which the flow depends on the velocity of the pulled out piston face.

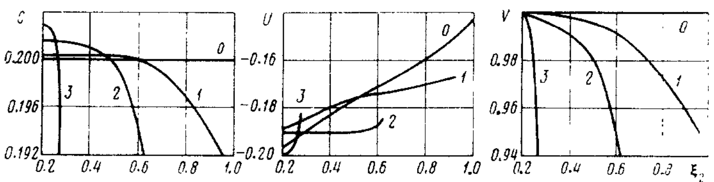

Fig.2

Since in regions 2 and 4 that face of the piston does not affect the flow, it is possible to compare the results of difference calculation of the flow in region 4 with the approximate solution (2).

The plane unsteady flow induced by an angular piston is defined by the system of equations

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x_1} + \frac{\partial G}{\partial x_2} = 0; \quad m = \rho u_1, \quad n = \rho u_2$$

$$F = \left\{m, \frac{m^2}{\rho} + P, \frac{mn}{\rho}, \frac{(e + P)\,m}{\rho}\right\}$$

$$G = \left\{n, \frac{mn}{\rho}, \frac{n^2}{\rho} + P, \frac{(e + P)\,n}{\rho}\right\}, \quad P = (\gamma - 1)\left(e - \frac{m^2 + n^2}{2\rho}\right)$$

where e is the total energy, P is the pressure, and W are the sought quantities.

Computation were carried out using the algorithm (*) conforming to the McCormack difference scheme of rip-through calculation of second order accuracy with nonlinear matched smoothing.

Owing to the program peculiarities computations did not begin from $t = 0$ but from some small $t_* > 0$. They had shown stability of the flow pattern for various t_*. A simple Riemann wave was taken as the initial conditions. The time step was selected on the basis of stability condition for the considered scheme ($\Delta t/\Delta h < 0.2$, $\Delta t = 0.01$ and $\Delta t = 0.005$), with the number of grid nodes 41×91. The flow represented in Fig.3 is for $t = 0.3-0.4$ and computations were carried out up to $t = 1.2-1.4$. The boundary conditions were specified on the dash-dot lines

*) A.I. Zhmakin and A.A. Fursenko, On a class of single-point difference schemes of rip-through calculation. Preprint No.623. Fiz. Tekhn. Inst. Leningrad, 1979.

(Fig.3) by values of parameters in region $1, 2, 3, 5$, and at piston faces conditions of impenetrability were specified.

Position of the curvilinear shock wave generated by the piston motion at velocity $U_L = (-1.5; 0.1), \gamma = 1.4$ is shown in Fig.4. Curves $1-3$ correspond to $t = 0.4; 0.8; 1$. Width of the shock wave zone 1 – 3 is determined by the pitch of the difference grid of the problem. For comparison a section of the shock wave determined by the approximate solution of $\xi_2 = f_0(\xi_1)$ is shown there by the heavy continuous line. It lies in the zone 3 of shock wave "blurring". The self similarity of the whole flow is confirmed by the position of shock waves 1 – 3 in the plane of variables $\xi_1 = x_1/t, \xi_2 = x_2/t$ where they all merge into one line. The dash lines in Fig. 4 represent lateral shock waves which, although weak, are reasonably clearly discernible; they coincide with $x_1 = t$ (the boundary between regions 3 and 4) and with $x_1 = (1 + a(\gamma + 1)/2)t$, $a = -1.5$ (the boundary between 4 and 2). Position of the contact discontinuity which has not separately taken into account in the derivation of the approximate solution, was also determined by difference computation. Contact discontinuities 4 – 6 correspond to curvilinear shock waves 1 – 3. For $t = 1$ the gasdynamic quantities in region 4 (Fig.4) determined by the difference computation method are in good agreement with the approximate solution of (2). The different problem was calculated for

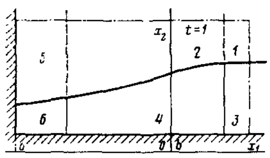

Fig.3

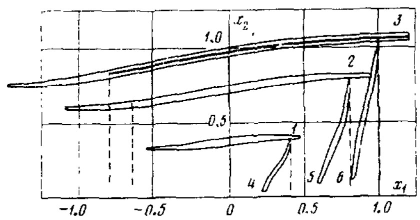

Fig.4

$$U_L = (-5/6;\ 0.1),\ (-1.5;\ 0.1),\ (-2;\ 0.1),\ (-1.5;\ 0.5),\ (-2;\ 1),\ (-5/6;\ 1)$$

up to $t = 1.4$ with $\gamma = 1.4$.

The variant $\gamma = 1.8$, $a = -2.9$, $b = 1$ was also calculated. In it density in region 5 is close to zero ($\rho = 0.001$) and the flow shown in Fig.1 is nearly isobaric. It is interesting to establish the boundary between the flow in region 4 (Fig.1) and the vacuum, whether attain the curvilinear shock wave $\xi_2 = f(\xi_1)$ to the vacuum boundary and gas $\xi_1 = -1/\varkappa$, whether in the piston face neighborhood $\xi_1 = b$ owing to additional pressure "tongues" of gas, issuing to the left beyond $\xi_1 = -1/\varkappa$ are formed. Calculations show that the curvilinear shock wave touches the piston face $x_2 = bt$, when at the unperturbed Riemann wave boundary $x_1 \approx -t/\varkappa$ and vacuum.

The results of difference computations enable us to make the following conclusions.

1) The qualitative pattern of flow, appearing in Fig.3 and coinciding with the approximate solution (with small b), and the self-similarity of solution remains valid throughout the considered range of variation of a and b for $1 < \gamma < 3$, $-2.9 < a < 0$, $0 < b < 1$.

2) The curvilinear shock wave $\xi_2 = f(\xi_1)$ finishes on the piston face $x_1 = bt$, and at the vacuum-gas interface there are no gas "tongues".

3) The approximate solution can be used up to $b = 0.2$—0.3 outside the neighborhood of the gas-vacuum interface.

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